Linear Weingarten surfaces in Euclidean and hyperbolic space

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This paper is dedicated to Manfredo do Carmo in admiration for his mathematical achievements and his influence on the field of differential geometry of surfaces

Abstract

In this paper we review some author's results about Weingarten surfaces in Euclidean space $\mathbb{R}^3$ and hyperbolic space $\mathbb{H}^3$. We stress here in the search of examples of linear Weingarten surfaces that satisfy a certain geometric property. First, we consider Weingarten surfaces in $\mathbb{R}^3$ that are foliated by circles, proving that the surface is rotational, a Riemann example or a generalized cone. Next we classify rotational surfaces in $\mathbb{R}^3$ of hyperbolic type showing that there exist surfaces that are complete. Finally, we study linear Weingarten surfaces in $\mathbb{H}^3$ that are invariant by a group of parabolic isometries, obtaining its classification.

MSC: 53C40, 53C50

Keywords: Weingarten surface; cyclic surface; Riemann examples; parabolic surface.

1 Statement of results

A surface S in Euclidean space $\mathbb{R}^3$ or hyperbolic space $\mathbb{H}^3$ is called a *Weingarten surface* if there is some smooth relation $W(\kappa_1, \kappa_2) = 0$ between its two principal curvatures κ_1 and κ_2. In particular, if K and H denote respectively the Gauss curvature and the mean curvature of S, $W(\kappa_1, \kappa_2) = 0$ implies a relation $U(K, H) = 0$. The classification of Weingarten surfaces in the general case is almost completely open today. After earlier works in the fifties due to Chern, Hopf, Voss, Hartman, Winter, amongst others, there has been recently a progress in this theory, specially when the Weingarten relation is of type $H = f(H^2 - K)$ and f elliptic. In such case, the surfaces satisfy a maximum principle that allows a best knowledge of the shape of such surfaces. These achievements can see, for example, in [2, 4, 5, 14, 15, 16].

The simplest case of functions W or U is that they are linear, that is,

$$a\kappa_1 + b\kappa_2 = c \qquad \text{or} \qquad aH + bK = c, \tag{1}$$

where a, b and c are constant. Such surfaces are called *linear Weingarten* surfaces. Typical examples of linear Weingarten surfaces are umbilical surfaces, surfaces with constant Gauss curvature and surfaces with constant mean curvature.

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A first purpose of the present work is to provide examples of linear Weingarten surfaces that satisfy a certain geometric condition. A first attempt is to consider that the surface is rotational, that is, invariant by a group of isometries that leave fixed-pointwise a geodesic of the ambient space. In such case, equations (1) lead to an ordinary differential equations and the study is then reduced to finding the profile curve that defines the surface.

A more general family of rotational surfaces are the cyclic surfaces, which were introduced by Enneper in the XIX century. A *cyclic surface* in Euclidean space $\mathbb{R}^3$ or $\mathbb{H}^3$ is a surface determined by a smooth uniparametric family of circles. Thus, a cyclic surface S is a surface foliated by circles meaning that there is a one-parameter family of planes which meet S in these circles. The planes are not assumed parallel, and if two circles should lie in planes that happen to be parallel, the circles are not assumed coaxial. Rotational surfaces are examples of cyclic surfaces.

Our first result is motivated by what happens for cyclic surfaces with constant mean curvature H. Recall that the catenoid is the only minimal ($H = 0$) rotational surface in $\mathbb{R}^3$. If the surface is not rotational, then the only cyclic minimal surfaces are a family of examples of periodic minimal surfaces discovered by Riemann, usually called in the literature as Riemann examples [13]. If the mean curvature H is a non-zero constant, then the only cyclic surfaces are the surfaces of revolution (Delaunay surfaces) [12]. In order to find new examples of linear Weingarten surfaces, we pose the following question: do exist non-rotational cyclic surfaces that are linear Weingarten surfaces?

In Section 2 we prove the following result:

Theorem 1.1. *Let S be a cyclic surface in Euclidean space $\mathbb{R}^3$.*

1. *If S satisfies a relation of type $a\kappa_1 + b\kappa_2 = c$, then S is a surface of revolution or it is a Riemann example ($H = 0$).*
2. *If S satisfies a relation of type $aH + bK = c$, then S is a surface of revolution or it is a Riemann example ($H = 0$) or it is a generalized cone ($K = 0$).*

Recall that a generalized cone is a cyclic surface formed by a uniparametric family of u-circles whose centres lie in a straight line and the radius function is linear on u. These surfaces have $K \equiv 0$ and they are the only non-rotational cyclic surfaces in $\mathbb{R}^3$ with constant Gaussian curvature [6].

After Theorem 1.1, we focus on Weingarten surfaces of revolution in $\mathbb{R}^3$. The classification of linear Weingarten surfaces strongly depends on the sign of $\Delta := a^2 + 4bc$. If $\Delta > 0$, the surface is said elliptic and satisfies good properties, as for example, a maximum principle: see [5, 14]. If $\Delta = 0$, the surface is a tube, that is, a cyclic surface where the circles have the same radius. Finally, if $\Delta < 0$, the surface is said *hyperbolic* (see [1]). In Section 3 we study hyperbolic rotational surfaces in $\mathbb{R}^3$. We do an explicit description of the hyperbolic rotational linear Weingarten. Examples of hyperbolic Weingarten surfaces are the surfaces with constant negative Gaussian curvature K: we take $a = 0$, $b = 1$ and $c < 0$ in the right relation (1). In contrast to the Hilbert's theorem that asserts that do not exist complete surfaces with constant negative Gaussian curvature immersed in $\mathbb{R}^3$, we obtain (see Theorem 3.5):

Theorem 1.2. *There exists a family of hyperbolic linear Weingarten complete rotational surfaces in $\mathbb{R}^3$ that are non-embedded and periodic.*

Finally, we are interested in linear Weingarten surfaces of revolution in hyperbolic space $\mathbb{H}^3$. In hyperbolic space there exist three types of rotational surfaces. We will study one of them, called

parabolic surfaces, that is, surfaces invariant by a group of parabolic isometries of the ambient space. This was began by do Carmo and Dajczer in the study of rotational surfaces in $\mathbb{H}^3$ with constant curvature [3] and works of Gomes, Leite, Mori et al. We will consider problems such as existence, symmetry and behaviour at infinity. As a consequence of our work, we obtain the following

Theorem 1.3. *There exist parabolic complete surfaces in $\mathbb{H}^3$ that satisfy the relation $aH+bK=c$.*

Part of the results of this work have recently appeared in a series of author's papers: [7, 8, 9, 11].

2 Cyclic Weingarten surfaces in $\mathbb{R}^3$

In this section we will study linear Weingarten surfaces that are foliated by a uniparametric family of circles (cyclic surfaces). In order to show the techniques to get Theorem 1.1, we only consider Weingarten surfaces that satisfy the linear relation $aH + bK = c$. The proof consists into two steps. First, we prove

Theorem 2.1. *Let S be a surface that satisfies $aH + bK = c$ and it is foliated by circles lying in a one-parameter family of planes. Then either S is a subset of a round sphere or the planes of the foliation are parallel.*

Proof. Consider $P(u)$ the set of planes of the foliation, that is, $S = \bigcup_{u\in I} P(u)\cap S$, $u \in I \subset \mathbb{R}$, and such that $P(u)\cap S$ is a circle for each u. Assume that the planes $P(u)$ are not parallel . Then we are going to show that the surfaces is a sphere. The proof follows the same ideas for the case of the constancy of the mean curvature [12]. Let Γ be an orthogonal curve to the foliation planes, that is, $\Gamma'(u)\perp P(u)$. If $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ denotes the usual Frenet trihedron of Γ, the surface S is locally parametrized by

$$X(u,v) = \mathbf{c}(u) + r(u)(\cos v\ \mathbf{n}(u) + \sin v\ \mathbf{b}(u)), \tag{2}$$

where $r = r(u) > 0$ and $\mathbf{c} = \mathbf{c}(u)$ denote respectively the radius and centre of each circle $P(u)\cap S$. We compute the mean curvature and the Gauss curvature of S using the usual local formulae

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}, \qquad K = \frac{eg - f^2}{EG - F^2}.$$

Here $\{E, F, G\}$ and $\{e, f, g\}$ represent the coefficients of the first and second fundamental form, respectively. Then the relation $aH + bK = c$ writes in terms of the curve Γ. Using the Frenet equations of Γ, we are able to express the relation $aH + bK = c$ as a trigonometric polynomial on $\cos(nv)$ and $\sin(nv)$:

$$A_0 + \sum_{n=1}^{8}\Big(A_n(u)\cos(nv) + B_n(u)\sin(nv)\Big) = 0, \qquad u \in I, v \in [0, 2\pi].$$

Here A_n and B_n are smooth functions on u. Because the functions $\cos(nv)$ and $\sin(nv)$ are independent, all coefficient functions A_n, B_n must be zero. This leads to a set of equations, which we wish to solve. Because the curve Γ is not a straight line, its curvature κ does not vanish.

The proof consists into the explicit computation of the coefficients A_n and B_n and solving $A_n = B_n = 0$. The proof program begins with the equations $A_8 = 0$ and $B_8 = 0$, which yields relations between the geometric quantities of the curve Γ. By using these data, we follow with equations $A_7 = B_7 = 0$ and so on, until to arrive with $n = 0$. The author was able to obtain the results using the symbolic program Mathematica to check his work: the computer was used in each calculation several times, giving understandable expressions of the coefficients A_n and B_n. Finally, we achieve to show that X is a parametrization of a round sphere.

□

Once proved Theorem 2.1, the following step consists to conclude that either the circles of the foliation must be coaxial (and the surface is rotational) or that $K \equiv 0$ or $H \equiv 0$. In the latter cases, the Weingarten relation (1) is trivial in the sense that $a = c = 0$ or $b = c = 0$.

Theorem 2.2. *Let S be a cyclic surface that satisfies $aH + bK = c$. If the foliation planes are parallel, then either S is a surface of revolution or $a = c = 0$ or $b = c = 0$.*

Proof. After an isometry of the ambient space, we parametrize S as

$$X(u, v) = (f(u), g(u), u) + r(u)(\cos v, \sin v, 0),$$

where f, g and r are smooth functions on u, $u \in I \subset \mathbb{R}$ and $r(u) > 0$ denotes the radius of each circle of the foliation. Then S is a surface of revolution if and only if f y g are constant functions. Proceeding similarly as in the proof of Theorem 2.1, Equation $aH + bK = c$ is equivalent to an expression

$$\sum_{n=0}^{8} \Big(A_n(u) \cos(nv) + B_n(u) \sin(nv) \Big) = 0.$$

Again, the functions A_n and B_n must vanish on I. Assuming that the surface is not rotational, that is, $f'(u)g'(u) \neq 0$ at some point u, we conclude that $a = c = 0$ or $b = c = 0$. □

Recall what happens in the latter cases. The computation of $H \equiv 0$ and $K \equiv 0$ gives

$$f'' = \lambda r^2, \qquad g'' = \mu r^2, \qquad 1 + (\lambda^2 + \mu^2) r^4 + r'^2 - r r'' = 0, \tag{3}$$

and

$$f'' = g'' = r'' = 0, \tag{4}$$

respectively. If (3) holds, we have the equations that describe the Riemann examples ($\lambda^2 + \mu^2 \neq 0$) and the catenoid ($\lambda = \mu = 0$). In the case (4), the surface S is a generalized cone.

As a consequence of the above Theorems 2.1 and 2.2, we obtain Theorem 1.1 announced in the introduction of this work. Finally, the previous results allow us to give a characterization of Riemann examples and generalized cones in the class of linear Weingarten surfaces.

Corollary 2.3. *Riemann examples and generalized cones are the only non-rotational cyclic surfaces that satisfy a Weingarten relation of type $aH + bK = c$.*

3 Hyperbolic linear Weingarten surfaces in $\mathbb{R}^3$

We consider surfaces S in Euclidean space that satisfy the relation

$$a\ H + b\ K = c \tag{5}$$

where a, b and c are constants under the relation $a^2 + 4bc < 0$. These surfaces are called *hyperbolic linear Weingarten surfaces*. In particular, $c \neq 0$, which can be assumed to be $c = 1$. Thus the condition $\Delta < 0$ writes now as $a^2 + 4b < 0$. In this section, we study these surfaces in the class of surfaces of revolution. Equation (5) leads to an ordinary differential equation that describes the generating curve α of the surface. Without loss of generality, we assume S is a rotational surface whose axis is the x-axis. If $\alpha(s) = (x(s), 0, z(s))$ is arc-length parametrized and the surface is given by $X(s,\phi) = (x(s), z(s)\cos\phi, z(s)\sin\phi)$, then (5) leads to

$$a\ \frac{\cos\theta(s) - z(s)\theta'(s)}{2z(s)} - b\ \frac{\cos\theta(s)\theta'(s)}{z(s)} = 1, \tag{6}$$

where $\theta = \theta(s)$ the angle function that makes the velocity $\alpha'(s)$ at s with the x-axis, that is, $\alpha'(s) = (\cos\theta(s), 0, \sin\theta(s))$. The curvature of the planar curve α is given by θ'. In this section, we discard the trivial cases in (5), that is, $a = 0$ (constant Gauss curvature) and $b = 0$ (constant mean curvature).

The generating curve α is then described by the solutions of the O.D.E.

$$\left\{\begin{array}{lcl} x'(s) & = & \cos\theta(s) \\ z'(s) & = & \sin\theta(s) \\ \theta'(s) & = & \dfrac{a\cos\theta(s) - 2z(s)}{az(s) + 2b\cos\theta(s)} \end{array}\right. \tag{7}$$

Assume initial conditions

$$x(0) = 0, \qquad z(0) = z_0, \qquad \theta(0) = 0. \tag{8}$$

Without loss of generality, we can choose the parameters a and z_0 to have the same sign: in our case, we take to be positive numbers.

A first integral of (7)-(8) is given by

$$z(s)^2 - az(s)\cos\theta(s) - b\cos^2\theta(s) - (z_0^2 - az_0 - b) = 0. \tag{9}$$

By the uniqueness of solutions, any solution $\alpha(s) = (x(s), 0, z(s))$ of (7)-(8) is symmetric with respect to the line $x = 0$.

In view of (8), the value of $\theta'(s)$ at $s = 0$ is $\theta'(0) = \dfrac{a - 2z_0}{az_0 + 2b}$. Our study depends on the sign of $\theta'(0)$. We only consider the case

$$z_0 > \frac{-2b}{a} \tag{10}$$

which implies that $z_0 > a/2$. The denominator in the third equation of (7) is positive since it does not vanish and at $s = 0$, its value is $az_0 + 2b > 0$. As $z_0 > a/2$, the numerator in (7) is negative. Thus we conclude that the function $\theta'(s)$ is negative anywhere.

From (9), we write the function $z = z(s)$ as

$$z(s) = \frac{1}{2}\left(a\cos\theta(s) + \sqrt{(a^2+4b)\cos^2\theta(s) + 4(z_0^2 - az_0 - b)}\right). \tag{11}$$

Lemma 3.1. *The maximal interval of the solution (x, z, θ) of (7)-(8) is $\mathbb{R}$.*

Proof. The result follows if we prove that the derivatives x', z' and θ' are bounded. In view of (7), it suffices to show it for θ' (recall that $\theta'(s) < 0$). We are going to find a negative number m such that $m \leq \theta'(s)$ for all s. To be exact, we show the existence of constants δ and η independent on s, with $\eta < 0 < \delta$, such that

$$az(s) + 2b\cos\theta(s) \geq \delta \quad \text{ and } \quad a\cos\theta(s) - 2z(s) \geq \eta. \tag{12}$$

Once proved this, it follows from (7) that

$$\theta'(s) \geq \frac{\eta}{\delta} := m. \tag{13}$$

Define the function $f(z_0) := z_0^2 - az_0 - b$. The function f is strictly increasing on z_0 for $z_0 > a/2$. Using that $a^2 + 4b < 0$, we have $\frac{a}{2} < \frac{-2b}{a}$. As z_0 satisfies (10), there exists $\epsilon > 0$ such that

$$z_0^2 - az_0 - b = f(-\frac{2b}{a}) + \epsilon = \frac{b(a^2+4b)}{a^2} + \epsilon.$$

From (11),

$$z(s) \geq \frac{1}{2}\left(a\cos\theta + \sqrt{(a^2+4b)\cos^2\theta + \frac{4b}{a^2}(a^2+4b) + 4\epsilon}\right) \geq \frac{1}{2}\left(a\cos\theta - \frac{a^2+4b}{a} + \epsilon'\right),$$

for a certain positive number ϵ'. By using that $a^2 + 4b < 0$ again, we have

$$az(s) + 2b\cos\theta(s) \geq \frac{a^2+4b}{2}(\cos\theta(s) - 1) + \frac{a}{2}\epsilon' \geq \frac{a}{2}\epsilon' := \delta.$$

By using (11) again, we obtain

$$a\cos\theta(s) - 2z(s) \geq -\sqrt{(a^2+4b)\cos^2(s)\theta + 4f(z_0)} \geq -2\sqrt{f(z_0)} := \eta,$$

which concludes the proof of this lemma. $\square$

Lemma 3.2. *For each solution (x, z, θ) of (7)-(8), there exists $M < 0$ such that $\theta'(s) < M$.*

Proof. It suffices if we prove that there exist δ_2, η_2, with $\eta_2 < 0 < \delta_2$ such that

$$az(s) + 2b\cos\theta(s) \leq \delta_2 \quad \text{ and } \quad a\cos\theta(s) - 2z(s) \leq \eta_2,$$

since (7) yields $\theta'(s) \leq \delta_2/\eta_2 := M$. Using (11), we have

$$\begin{aligned} az(s) + 2b\cos\theta(s) &= \frac{1}{2}\left((a^2+4b)\cos\theta(s) + a\sqrt{(a^2+4b)\cos^2\theta(s) + 4f(z_0)}\right) \\ &\leq a\sqrt{f(z_0)} := \delta_2. \end{aligned}$$

On the other hand,

$$\begin{aligned} a\cos\theta(s) - 2z(s) &= -\sqrt{(a^2+4b)\cos^2\theta(s)+4f(z_0)} \\ &\leq -\sqrt{(a^2+4b)+4f(-2b/a)} := \eta_2. \end{aligned}$$

□

Lemma 3.2 implies that $\theta(s)$ is strictly decreasing with

$$\lim_{s\to\infty}\theta(s) = -\infty.$$

Since Lemma 3.1 asserts that any solution is defined for any s, put $T > 0$ the first number such that $\theta(T) = -2\pi$. We prove that α is a periodic curve.

Lemma 3.3. *Under the hypothesis of this section and with the above notation, we have:*

$$\begin{aligned} x(s+T) &= x(s) + x(T) \\ z(s+T) &= z(s) \\ \theta(s+T) &= \theta(s) - 2\pi \end{aligned}$$

Proof. This is a consequence of the uniqueness of solutions of (7)-(8). We only have to show that $z(T) = z_0$. But this is a direct consequence of the assumption (10), that $a^2 + 4b < 0$ and (11). □

As conclusion of Lemma 3.3, we describe the behavior of the coordinates functions of the profile curve α under the hypothesis (10). Due to the monotonicity of θ, let T_1, T_2 and T_3 be the points in the interval $[0, T]$ such that the function θ takes the values $-\pi/2, -\pi$ and $-3\pi/2$ respectively. In view of the variation of the angle θ with the time coordinate s, it is easy to verify the following Table:

s	θ	$x(s)$	$z(s)$
$[0, T_1]$	$[\frac{-\pi}{2}, 0]$	increasing	decreasing
$[T_1, T_2]$	$[-\pi, \frac{-\pi}{2}]$	decreasing	decreasing
$[T_2, T_3]$	$[\frac{-3\pi}{2}, -\pi]$	decreasing	increasing
$[T_3, T]$	$[-2\pi, \frac{-3\pi}{2}]$	increasing	increasing

Theorem 3.4. *Let $\alpha = \alpha(s) = (x(s), 0, z(s))$ be the profile curve of a rotational hyperbolic surface S in $\mathbb{R}^3$ where α is the solution of (7)-(8). Assume that the initial condition on z_0 satisfies $z_0 > \frac{-2b}{a}$. Then (see Fig. 1)*

1. *The curve α is invariant by the group of translations in the x-direction given by the vector $(x(T), 0, 0)$.*
2. *In the period $[0, T]$ of z given by Lemma 3.3, the function $z = z(s)$ presents one maximum at $s = 0$ and one minimum at $s = T_2$. Moreover, α is symmetric with respect to the vertical line at $x = 0$ and $x = x(T_2)$.*

3. *The height function of α, that is, $z = z(s)$, is periodic.*

4. *The curve α has self-intersections and its curvature has constant sign.*

5. *The part of α between the maximum and the minimum satisfies that the function $z(s)$ is strictly decreasing with exactly one vertical point. Between this minimum and the next maximum, $z = z(s)$ is strictly increasing with exactly one vertical point.*

6. *The velocity α' turns around the origin.*

Theorem 3.5. *Let S be a rotational hyperbolic surface in $\mathbb{R}^3$ whose profile curve α satisfies the hypothesis of Theorem 3.4. Then S has the following properties:*

1. *The surface has self-intersections.*

2. *The surface is periodic with infinite vertical symmetries.*

3. *The surface is complete.*

4. *The part of α between two consecutive vertical points and containing a maximum corresponds with points of S with positive Gaussian curvature; on the other hand, if this part contains a minimum, the Gaussian curvature is negative.*

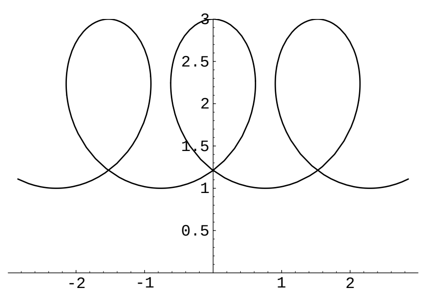


Figure 1: The generating curve of a rotational hyperbolic surfaces, with $a = -b = 2$. Here $z_0 = 3$. The curve α is periodic with self-intersections.

As it was announced in Theorem 1.2, and in order to distinguish from the surfaces of negative constant Gaussian curvature, we conclude from Theorem 3.5 the following

Corollary 3.6. *There exists a one-parameter family of rotational hyperbolic linear Weingarten surfaces that are complete and with self-intersections in $\mathbb{R}^3$. Moreover, the generating curves of these surfaces are periodic.*

4 Parabolic Weingarten surfaces in $\mathbb{H}^3$

A parabolic group of isometries of hyperbolic space $\mathbb{H}^3$ is formed by isometries that leave fix one double point of the ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$. A surface S in $\mathbb{H}^3$ is called a *parabolic surface* if it is

invariant by a group of parabolic isometries. A parabolic surface S is determined by a generating curve α obtained by the intersection of S with any geodesic plane orthogonal to the orbits of the group.

We consider the upper half-space model of $\mathbb{H}^3$, namely, $\mathbb{H}^3 =: \mathbb{R}^3_+ = \{(x,y,z) \in \mathbb{R}^3; z > 0\}$ equipped with the metric $\langle , \rangle = \frac{dx^2+dy^2+dz^2}{z^2}$. The ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$ is identified with the one point compactification of the plane $\Pi \equiv \{z=0\}$, that is, $\mathbb{S}^2_\infty = \Pi \cup \{\infty\}$. In what follows, we will use the words vertical or horizontal in the usual affine sense of $\mathbb{R}^3_+$. Denote $L = \mathbb{S}^2_\infty \cap \{y=0\}$.

Let G be a parabolic group of isometries. In the upper half-space model, we take the point $\infty \in \mathbb{S}^2_\infty$ as the point that fixes G. Then the group G is defined by the horizontal (Euclidean) translations in the direction of a horizontal vector ξ with $\xi \in \Pi$ which can be assumed $\xi = (0,1,0)$.

A (parabolic) surface S invariant by G parametrizes as $X(s,t) = (x(s), t, z(s))$, where $t \in \mathbb{R}$ and the curve $\alpha = (x(s), 0, z(s))$, $s \in I \subset \mathbb{R}$, is assumed to be parametrized by the Euclidean arc-length. The curve α is the generating curve of S. We write $\alpha'(s) = (\cos\theta(s), 0, \sin\theta(s))$, for a certain differentiable function θ, where the derivative $\theta'(s)$ is the Euclidean curvature of α. With respect to the unit normal vector $N(s,t) = (-\sin\theta(s), 0, \cos\theta(s))$, the principal curvatures are

$$\kappa_1(s,t) = z(s)\theta'(s) + \cos\theta(s), \qquad \kappa_2(s,t) = \cos\theta(s).$$

The relation $aH + bK = c$ writes then

$$\left(\frac{a}{2} + b\cos\theta(s)\right) z(s)\theta'(s) + a\cos\theta(s) - b\sin^2\theta(s) = c. \tag{14}$$

We consider initial conditions

$$x(0) = 0, \qquad z(0) = z_0 > 0, \qquad \theta(0) = 0. \tag{15}$$

Then any solution $\{x(s), z(s), \theta(s)\}$ satisfies properties of symmetry which are consequence of the uniqueness of solutions of an O.D.E. For example, the solution is symmetric with respect to the vertical straight line $x = 0$. Using uniqueness again, we infer immediately

Proposition 4.1. *Let α be a solution of the initial value problem (14)-(15) with $\theta(0) = \theta_0$. If $\theta'(s_0) = 0$ at some real number s_0, then α is parameterized by $\alpha(s) = ((\cos\theta_0)s, 0, (\sin\theta_0)s + z_0)$, that is, α is a straight line and the corresponding surface is a totally geodesic plane, an equidistant surface or a horosphere.*

In view of this proposition, we can assume that the function $\theta'(s)$ do not vanish, that is, θ is a monotonic function on s. At $s = 0$, Equation (14) is

$$\theta'(0) = \frac{2}{z_0}\frac{c-a}{a+2b}.$$

This means that the study of solutions of (14)-(15) must analyze a variety of cases depending on the sign of $\theta'(0)$. In this section, we are going to consider some cases in order to show techniques and some results. First, assume that $c \neq 0$, which it can be assumed to be $c = 1$. Then we write (14) as

$$\theta'(s) = 2\,\frac{1 - a\cos\theta(s) + b\sin^2\theta(s)}{z(s)(a + 2b\cos\theta(s))}. \tag{16}$$

Our first result considers a case where it is possible to obtain explicit examples.

Theorem 4.2. *Let $\alpha(s) = (x(s), 0, z(s))$ be the generating curve of a parabolic surface S in hyperbolic space $\mathbb{H}^3$ that satisfies $aH+bK = 1$ with initial conditions (15). Assume $a^2+4b^2+4b = 0$. Then α describes an open of an Euclidean circle in the xz-plane.*

Proof. Equation (14) reduces into

$$-2bz(s)\theta'(s) = a + 2b\cos\theta(s).$$

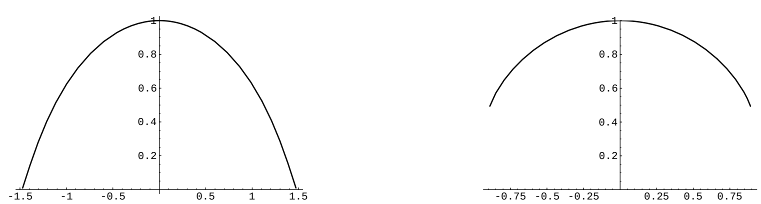
By differentiation with respect to s, we obtain $z(s)\theta''(s) = 0$, that is, $\theta'(s)$ is a constant function. Since $\theta'(s)$ describes the Euclidean curvature of α, we conclude that α parametrizes an Euclidean circle in the xz-plane and the assertion follows. This circle may not to be completely included in the halfspace $\mathbb{R}^3_+$. □

From now, we assume $a^2 + 4b^2 + 4b \neq 0$. Let us denote by $(-\bar{s}, \bar{s})$ the maximal domain of the solutions of (14)-(15). By the monotonicity of $\theta(s)$, let $\theta_1 = \lim_{s\to\bar{s}} \theta(s)$.

Theorem 4.3 (Case $0 < a < 1$)**.** *Let $\alpha(s) = (x(s), 0, z(s))$ be the generating curve of a parabolic surface S in hyperbolic space $\mathbb{H}^3$ that satisfies $aH + bK = 1$ with initial conditions (15). Assume $b \neq 0$ and $0 < a < 1$.*

1. *If $a + 2b < 0$, α has one maximum and α is a concave (non-entire) vertical graph. If $b < -(1 + \sqrt{1 - a^2})/2$, the surface S is complete and intersects $\mathbb{S}^2_\infty$ making an angle θ_1 such that $2\cos\theta_1 - b\sin^2\theta_1 = 0$. The asymptotic boundary of S is formed by two parallel straight lines. See Fig. 2 (a). If $-(1 + \sqrt{1 - a^2})/2 < b < -a/2$, then S is not complete. See Fig. 2 (b).*

2. *Assume $a + 2b > 0$. If $a - 2b > 0$, then S is complete and invariant by a group of translations in the x-direction. Moreover, α has self-intersections and it presents one maximum and one minimum in each period. See Fig. 3, (a). If $a - 2b \leq 0$, then S is not complete. Moreover α is not a vertical graph with a minimum. See Fig. 3, (b).*



(a) (b)

Figure 2: The generating curve of a parabolic surface with $aH + bK = 1$, with $0 < a < 1$ and $a + 2b < 0$. Here $z_0 = 1$ and $a = 0.5$. In the case (a), $b = -1$ and in the case (b), $b = -0.8$.

We point out that in each one of the cases of Theorem 4.3, we assert the existence of parabolic complete surfaces in $\mathbb{H}^3$ with the property $aH + bK = c$, such as it was announced in Theorem 1.3.

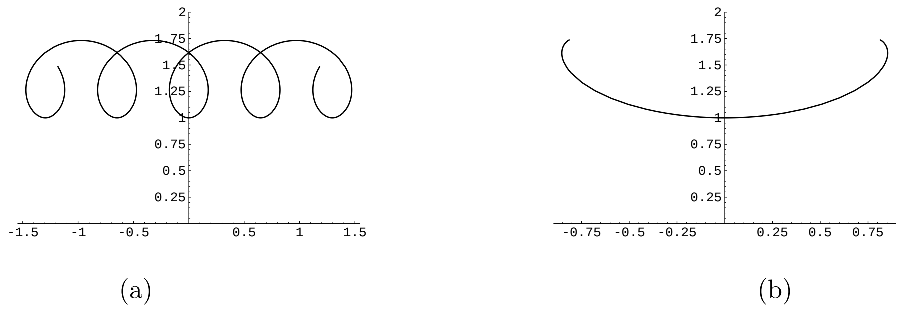


Figure 3: The generating curve of a parabolic surface with $aH + bK = 1$, with $0 < a < 1$ and $a + 2b > 0$. Here $z_0 = 1$ and $a = 0.5$. In the case (a), $b = -0.2$ and in the case (b), $b = 0.3$.

Proof. The second derivative of $\theta''(s)$ satisfies

$$-\theta'(s)\sin\theta(s)\Big[b\theta'(s) + \Big(\frac{a}{2} + b\cos\theta(s)\Big)\Big] + \Big(\frac{a}{2} + b\cos\theta(s)\Big) z(s)\theta''(s) = 0. \tag{17}$$

1. Case $a + 2b < 0$. Then $\theta'(0) < 0$ and $\theta(s)$ is strictly decreasing. If $\cos\theta(s) = 0$ at some point s, then (14) gives $(a/2)z(s)\theta'(s) - b - 1 = 0$. Thus, if $b \geq -1$, $\cos\theta(s) \neq 0$ and $-\pi/2 < \theta(s) < \pi/2$. In the case that $b < -1$ and as $a + 2b\cos\theta(s) < 0$, it follows from (14) that $a\cos\theta(s) - b\sin^2\theta(s) - 1 < 0$ for any value of s, in particular, $\cos\theta(s) \neq 0$. This proves that $x'(s) = \cos\theta(s) \neq 0$ and so, α is a vertical graph on L. This graph is concave since $z''(s) = \theta'(s)\cos\theta(s) < 0$. Moreover, this implies that $\bar{s} < \infty$ since on the contrary, and as $z(s)$ is decreasing with $z(s) > 0$, we would have $z'(s) \to 0$, that is, $\theta(s) \to 0$: contradiction.

   For $s > 0$, $z'(s) = \sin\theta(s) < 0$ and $z(s)$ is strictly decreasing. Set $z(s) \to z(\bar{s}) \geq 0$. The two roots of $4b^2 + 4b + a^2 = 0$ on b are $b = -\frac{1}{2}(1 \pm \sqrt{1-a^2})$. Moreover, and from $a + 2b < 0$, we have

   $$-\frac{1}{2}(1 + \sqrt{1-a^2}) < \frac{-a}{2} < -\frac{1}{2}(1 - \sqrt{1-a^2}).$$

   (a) Subcase $b < -(1 + \sqrt{1-a^2})/2$. Under this assumption, $a^2 + 4b^2 + 4b > 0$. Since $a < 1$, we obtain

   $$a + 2b\cos\theta(s) < -\sqrt{a^2 + 4b^2 + 4b}. \tag{18}$$

   If $z(\bar{s}) > 0$, then $\lim_{s\to\bar{s}}\theta'(s) = -\infty$. In view of (16) we have $a + 2b\cos\theta(\bar{s}) = 0$: contradiction with (18). Hence, $z(\bar{s}) = 0$ and α intersects L with an angle θ_1 satisfying $a\cos\theta_1 - b\sin^2\theta_1 - 1 = 0$.

   (b) Subcase $-(1 + \sqrt{1-a^2})/2 < b < -a/2$. Now $a^2 + 4b^2 + 4b < 0$. The function $1 - a\cos\theta(s) + b\sin^2\theta(s)$ is strictly decreasing and its value at $\bar{s}$ satisfies $\cos\theta(s) > -a/2b$. Thus

   $$1 - a\cos\theta(s) + b\sin^2\theta(s) \geq \frac{a^2 + 4b^2 + 4b}{4b} > 0. \tag{19}$$

   Assume $z(\bar{s}) = 0$. Then (16) and (19) imply that $\theta'(\bar{s}) = -\infty$. Combining (16) and (17), we have

   $$\frac{\theta''(s)}{\theta'(s)^2} = \frac{b\sin\theta(s)}{z(s)\left(\frac{a}{2} + b\cos\theta(s)\right)} + \frac{\sin\theta(s)\left(\frac{a}{2} + b\cos\theta(s)\right)}{1 - a\cos\theta(s) + b\sin^2\theta(s)}.$$

From this expression and as $\sin\theta(\bar{s}) \neq 0$, we conclude

$$\lim_{s\to\bar{s}} \frac{\theta''(s)}{\theta'(s)^2} = -\infty.$$

On the other hand, using L'Hôpital rule, we have

$$\lim_{s\to\bar{s}} z(s)\theta'(s) = \lim_{s\to\bar{s}} -\frac{\sin\theta(s)}{\frac{\theta''(s)}{\theta'(s)^2}} = 0.$$

By letting $s \to \bar{s}$ in (16), we obtain a contradiction. Thus, $z(\bar{s}) > 0$. This means that $\lim_{s\to\bar{s}} \theta'(s) = -\infty$ and it follows from (14) that

$$\lim_{s\to\bar{s}} \left(\frac{a}{2} + b\cos\theta(s)\right) = 0.$$

2. Case $a + 2b > 0$. Then $\theta'(0) > 0$ and $\theta(s)$ is strictly increasing. We distinguish two possibilities:

   (a) Subcase $a - 2b > 0$. We prove that $\theta(s)$ reaches the value π. On the contrary, $\theta(s) < \pi$ and $z(s)$ is an increasing function. The hypothesis $a-2b > 0$ together $a+2b > 0$ implies that $a + 2b\cos\theta(s) \geq \delta > 0$ for some number δ. From (16), $\theta'(s)$ is bounded and then $\bar{s} = \infty$. In particular, $\lim_{s\to\infty} \theta'(s) = 0$. As both $a-2b$ and $a+2b$ are positive numbers, the function $b\theta'(s) + (a + 2b\cos\theta(s))$ is positive near $\bar{s} = \infty$. Then using (17), $\theta''(s)$ is positive for a certain value of s big enough, which it is impossible. As conclusion, $\theta(s)$ reaches the value π at some $s = s_0$. By the symmetry properties of solutions of (14), α is symmetric with respect to the line $x = x(s_0)$ and the velocity vector of α rotates until to the initial position. This means that α is invariant by a group of horizontal translations.

   (b) Subcase $a - 2b \leq 0$. As $\theta'(s) > 0$, Equation (16) says that $\cos\theta(s) \neq -1$, and so, $\theta(s)$ is bounded by $-\pi < \theta(s) < \pi$. As in the above subcase, if $\bar{s} = \infty$, then $\theta'(s) \to 0$, and this is a contradiction. Then $\bar{s} < \infty$ and $\lim_{s\to\bar{s}} \theta'(s) = \infty$. Hence, $\cos\theta(\bar{s}) = -a/(2b)$ and $\theta(s)$ reaches the value $\pi/2$.

$\square$

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